Probability as a measure of *information-added*

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I want to turn this way of conceiving the relationship beween probability and information on its head. I plan to start from qualitative considerations on the information added by a proposition α to a body of propositions Γ, and hence to obtain quantitative measures of information.

We find that there are at least two viable notions of information-added: one goes by the *novelty value* of the added information, the other, very roughly, by the (weighted) proportion of consequences left open by the body of propositions that the added proposition rules out—this is information-added as *an additional resource in drawing consequences*.

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Making the sort of plenitude assumptions common in the area of representation theorems, we find that any *precise* measure of information-added as novelty value can be rescaled as a probability-*like* function and any measure of information-added as additional resource rescales as a unique Popper function. (What follows borrows heavily from the literature on quantitative measures of comparative conditional probability.)

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Vagueness of information-added is accommodated by working backwards from the family of precise sharpenings of an inexact relation of information-added.

Our starting point is the thought that given a sentence α and a collection of sentences Γ, α adds information to Γ: in general the set $\Gamma \cup \{\alpha\}$ is more informative than Γ, an obvious exception being when Γ entails α, for then the content of α is already contained in Γ

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More particularly, in what ways are $\preccurlyeq$ and $\prec$ responsive to logical structure and relations?

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These, I trust, are intuitively plausible.

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If no information can be added to $\Gamma \cup \{\alpha\}$, *i.e.* for all β, $\langle\beta, \Gamma \cup \{\alpha\}\rangle \approx \langle\alpha, \Gamma \cup \{\alpha\}\rangle$, then, $\langle\alpha, \Gamma\rangle \approx \langle\bot, \Gamma\rangle$, where $\bot$ is a contradiction.

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(vi$_2$) Provided there are propositions η and θ such that $\langle \eta, \Gamma \rangle \prec \langle \theta, \Gamma \rangle$, if $\langle \alpha, \Gamma \rangle \preccurlyeq \langle \beta, \Delta \rangle$ then $\langle \neg\beta, \Delta \rangle \preccurlyeq \langle \neg\alpha, \Gamma \rangle$.

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(vi$_2$) can be motivated by thinking of the information α adds to Γ in terms of the proportion of possibilities left open by Γ that are ruled out by α. The more possibilities that α rules out, the fewer its negation rules out and vice versa, a sentence adding more information to Γ the more possibilities it rules out.

The point is that these two conceptions of information-added, both quite viable, are incompatible. They are quite distinct conceptions. We see this because, taken together, they, along with (i) – (v) entail, for any α and Γ, that either α adds no information to Γ or it adds as much as a contradiction adds—the maximum possible.

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Proof.

If no information can be added to $\Gamma \cup \{\alpha\}$ then $\langle \alpha, \Gamma \rangle \approx \langle \bot, \Gamma \rangle$. Suppose, then, that information can be added to $\Gamma \cup \{\alpha\}$, *i.e.* that there are propositions η and θ such that $\langle \eta, \Gamma \cup \{\alpha\} \rangle \prec \langle \theta, \Gamma \cup \{\alpha\} \rangle$. Then, by (i), $\langle \alpha, \Gamma \cup \{\alpha\} \rangle \preccurlyeq \langle \alpha, \Gamma \rangle$, and, by ($\mathrm{vi}_1$), $\langle \neg\alpha, \Gamma \cup \{\alpha\} \rangle \preccurlyeq \langle \neg\alpha, \Gamma \rangle$. Applying ($\mathrm{vi}_2$) to the latter, $\langle \neg\neg\alpha, \Gamma \rangle \preccurlyeq \langle \neg\neg\alpha, \Gamma \cup \{\alpha\} \rangle$. As $\neg\neg\alpha \dashv\vdash \alpha$, by (iii), $\langle \alpha, \Gamma \rangle \preccurlyeq \langle \alpha, \Gamma \cup \{\alpha\} \rangle$. So $\langle \alpha, \Gamma \cup \{\alpha\} \rangle \approx \langle \alpha, \Gamma \rangle$. □

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(vi$_1$): $\langle \alpha, \Gamma \cup \{\alpha\}\rangle \preccurlyeq \langle \alpha, \Gamma\rangle \preccurlyeq \langle \bot, \Gamma\rangle \preccurlyeq \langle \bot, \emptyset\rangle$. By (ii), $\langle \alpha, \Gamma \cup \{\alpha\}\rangle \prec \langle \bot, \emptyset\rangle$.

From (vi$_2$) we get, if $\neg\neg\gamma \dashv\vdash \gamma$,

(vi$_2^*$) provided there are propositions η and θ such that $\langle \eta, \Gamma \rangle \prec \langle \theta, \Gamma \rangle$, if $\langle \alpha, \Gamma \rangle \prec \langle \beta, \Delta \rangle$ then $\langle \neg\beta, \Delta \rangle \prec \langle \neg\alpha, \Gamma \rangle$.

In different ways, (vi$_1$) and (vi$_2$) both establish the existence of maximal additions of information. In the case of (vi$_2$) there is a *uniform* maximum: the information added by a contradiction to any set of propositions to which information can be added.

Proof.

(vi$_1$): $\langle \alpha, \Gamma \cup \{\alpha\} \rangle \preccurlyeq \langle \alpha, \Gamma \rangle \preccurlyeq \langle \bot, \Gamma \rangle \preccurlyeq \langle \bot, \emptyset \rangle$. By (ii), $\langle \alpha, \Gamma \cup \{\alpha\} \rangle \prec \langle \bot, \emptyset \rangle$.

(vi$_2$): if $\exists \eta, \theta$ such that $\langle \eta, \Gamma \rangle \prec \langle \theta, \Gamma \rangle$, then, for all β and Δ, $\langle \neg\bot, \Gamma \rangle \preccurlyeq \langle \neg\beta, \Delta \rangle$, and so $\langle \beta, \Delta \rangle \approx \langle \neg\neg\beta, \Delta \rangle \preccurlyeq \langle \neg\neg\bot, \Gamma \rangle \approx \langle \bot, \Gamma \rangle$. □

Because $\perp \vdash \alpha$, for every α, even absent (vi$_2$), we have that $\langle \alpha, \Gamma \rangle \preccurlyeq \langle \perp, \Gamma \rangle$ and that $\langle \alpha, \Gamma \rangle \approx \langle \perp, \Gamma \rangle$ when no information can be added to $\Gamma \cup \{\alpha\}$.

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Because it will be useful to have this constraint later, forgetting about (vi$_2$), we add:

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Because it will be useful to have this constraint later, forgetting about (vi$_2$), we add:

(vi$_3$) $\langle\bot,\Gamma\rangle \preccurlyeq \langle\bot,\Delta\rangle$ for inconsistent $\bot$ and all Δ to which it is possible to add information.

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if $\langle \alpha, \Gamma \rangle \prec \langle \perp, \Gamma \rangle$ then $\langle \neg\alpha, \Gamma \rangle \approx \langle \perp, \Gamma \rangle$. — If α does not add to Γ as much as a contradiction does then $\neg\alpha$ does!

Refinements

(vii)(a) If $\langle \alpha \wedge \gamma, \Gamma \rangle \preccurlyeq \langle \beta \wedge \delta, \Delta \rangle$ and $\langle \beta, \Delta \rangle \preccurlyeq \langle \alpha, \Gamma \rangle$ then $\langle \gamma, \Gamma \cup \{\alpha\} \rangle \preccurlyeq \langle \delta, \Delta \cup \{\beta\} \rangle$ unless, perhaps, $\langle \beta, \Delta \rangle$ attains some maximum.

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(vii)(b) If $\langle \alpha \wedge \gamma, \Gamma \rangle \preccurlyeq \langle \beta \wedge \delta, \Delta \rangle$ and $\langle \delta, \Delta \cup \{\beta\} \rangle \preccurlyeq \langle \gamma, \Gamma \cup \{\alpha\} \rangle$ then $\langle \alpha, \Gamma \rangle \preccurlyeq \langle \beta, \Delta \rangle$ unless, perhaps, $\langle \delta, \Delta \cup \{\beta\} \rangle$ attains some maximum.

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These principles are more significant than may at first appear, for with them, assuming negation behaves classically, we can derive:

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(vii)(c) If $\langle \alpha \wedge \gamma, \Gamma \rangle \approx \langle \bot, \Gamma \rangle$ then $\langle \alpha, \Gamma \rangle \approx \langle \bot, \Gamma \rangle$ or $\langle \gamma, \Gamma \cup \{\alpha\} \rangle \approx \langle \bot, \Gamma \cup \{\alpha\} \rangle$.

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(vii)(d) if $\Gamma \vdash \neg(\alpha \wedge \gamma)$, $\Delta \vdash \neg(\beta \wedge \delta)$, $\langle \alpha, \Gamma \rangle \preccurlyeq \langle \beta, \Delta \rangle$ and $\langle \gamma, \Gamma \rangle \preccurlyeq \langle \delta, \Delta \rangle$ then $\langle \alpha \vee \gamma, \Gamma \rangle \preccurlyeq \langle \beta \vee \delta, \Delta \rangle$;

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(vii)(e) if for some ϵ and ζ, $\langle \epsilon, \Gamma \rangle \prec \langle \zeta, \Gamma \rangle$ and if $\Gamma \vdash \neg(\alpha \wedge \gamma)$, $\Delta \vdash \neg(\beta \wedge \delta)$, $\langle \alpha, \Gamma \rangle \prec \langle \beta, \Delta \rangle$ and $\langle \gamma, \Gamma \rangle \preccurlyeq \langle \delta, \Delta \rangle$ then $\langle \alpha \vee \gamma, \Gamma \rangle \prec \langle \beta \vee \delta, \Delta \rangle$.

So far, so good, but for all we know, it might be that for any α and consistent Γ,

$$\text{either } \langle\alpha, \Gamma\rangle \approx \langle\alpha, \Gamma \cup \{\alpha\}\rangle \text{ or } \langle\alpha, \Gamma\rangle \approx \langle\bot, \Gamma\rangle;$$

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We look at Bernard Koopman's approach.

Koopman has a straightforward way of securing a quantitative representation. He first adds a further principle, his *axiom of subdivision*, governing, as we employ it, information added:

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(viii$_{\mathrm{K}}$) For any propositions α_1, α_2, ..., α_n, β_1, β_2, ..., β_n and bodies of propositions Γ and Δ, if $\Gamma \vdash \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$, $\Gamma \vdash \neg(\alpha_i \wedge \alpha_j)$, $1 \leq i < j \leq n$, $\Delta \vdash \beta_1 \vee \beta_2 \vee \ldots \vee \beta_n$, $\Delta \vdash \neg(\beta_i \wedge \beta_j)$, $1 \leq i < j \leq n$, $\langle \alpha_1, \Gamma \rangle \preccurlyeq \langle \alpha_2, \Gamma \rangle \preccurlyeq \ldots \preccurlyeq \langle \alpha_n, \Gamma \rangle$ and $\langle \beta_1, \Delta \rangle \preccurlyeq \langle \beta_2, \Delta \rangle \preccurlyeq \ldots \preccurlyeq \langle \beta_n, \Delta \rangle$, then $\langle \alpha_1, \Gamma \rangle \preccurlyeq \langle \beta_n, \Delta \rangle$ —provided there are propositions η and θ such that $\langle \eta, \Delta \rangle \prec \langle \theta, \Delta \rangle$.

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Koopman's Plenitude Assumption For each $n \in \mathbb{N}^+$, there is a set of propositions Γ_n, and individual propositions α_1^n, α_2^n, ..., α_n^n, such that $\Gamma_n \vdash \alpha_1^n \vee \alpha_2^n \vee \ldots \vee \alpha_n^n$, $\Gamma_n \vdash \neg\left(\alpha_i^n \wedge \alpha_j^n\right)$, $1 \leq i < j \leq n$, and $\langle \alpha_1^n, \Gamma_n \rangle \approx \langle \alpha_2^n, \Gamma_n \rangle \approx \ldots \approx \langle \alpha_n^n, \Gamma_n \rangle \prec \langle \bot, \Gamma_n \rangle$.

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The Plenitude Assumption gives (viii$_{\mathrm{K}}$) something to work on.

From these and the other principles, (vi_1) *excluded*, Koopman is able to derive

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for any μ, m, ν and n, $1 \leq \mu \leq m$, $1 \leq \nu \leq n$,

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$$\text{Let } P(\beta, \Delta) = \sup\Big\{ q \in \mathbb{Q}\colon \text{ for some } m \in \mathbb{N}, \text{ for some } n \in \mathbb{N}^+, \\ q = \tfrac{m}{n} \text{ and } \langle \beta, \Delta \rangle \preccurlyeq \langle \alpha_1^n \vee \alpha_2^n \vee \ldots \vee \alpha_m^n, \Gamma_n \rangle \Big\}.$$

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Here $\langle \alpha_1^n \vee \alpha_2^n \vee \cdots \vee \alpha_m^n, \Gamma_n \rangle$ is $\langle \bot, \Gamma_n \rangle$ when $m = 0$.

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If we don't assume that $\preccurlyeq$ is connected, but do assume that *it can always be extended to a connected ordering*, we obtain a convex family of Popper functions. We can think of the extensions as *sharpenings* of the original ordering. This is how we accommodate *vague information*: the convex family yields an interval representation.

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Another possibility, giving us an additive measure, is to take

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There are, of course, countless other order-reversing bijections mapping $[0, 1]$ into some subset of $\mathbb{R}^+ \cup \{0, +\infty\}$ with 1 being mapped to 0. Nothing I have said encourages us to prefer any one to any other. The important fact is that the function P is recoverable from all of them.

Koopman has given us one way to get from qualitative constraints on information-added to a convex family of Popper functions representing the original qualitative ordering or to a unique Popper function when $\preccurlyeq$ is connected. It turns out that, under a plenitude assumption weaker than Koopman's, *any quantitative* measure of information added that is sensitive to the structural features listed above can be rescaled as an additive measure.

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